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A singularity-free solution for a charged fluid sphere in general relativity

K D Krori and Jayantimala Barua
Mathematical Physics Forum, Cotton College, Gauhati 781001, India

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Abstract. We have obtained here a singularity-free solution for a static charged fluid sphere in general relativity. The solution satisfies physical conditions inside the sphere.

1. Introduction

Recently Efinger (1965), Kyle and Martin (1967) and Wilson (1969) have found internal solutions for static charged spheres in general relativity, but none of these solutions is absolutely free from singularities. In Efinger's solution the metric has a singularity at the origin ($r = 0$). Solutions due to Kyle and Martin and Wilson do not have singularities at $r = 0$. But in both cases the metrics may have singularities at points other than the origin so that restrictions have to be imposed on the sphere to avoid them. They have dealt in detail in their respective papers with these possible singularities. We have found here from a simple analysis of the field equations a completely singularity-free solution for a static charged fluid sphere. The metric is regular everywhere and the pressure, mass density etc are finite throughout the sphere so that the solution satisfies physical conditions inside it.

2. The field equations and their solutions

We use here the spherically symmetric metric

$$ds^2 = -e^{\lambda}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + e^{\nu}\,dt^2. \tag{1}$$

The Einstein–Maxwell field equations are

$$R_\mu^\nu - \tfrac{1}{2}\delta_\mu^\nu R = -8\pi(M_\mu^\nu + E_\mu^\nu) \tag{2}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu = 4\pi\sigma V^\mu \tag{3}$$

$$F_{\mu\nu,\alpha} + F_{\nu\alpha,\mu} + F_{\alpha\mu,\nu} = 0 \tag{4}$$

with

$$M_\mu^\nu = (\rho + P)V^\nu V_\mu - g_\mu^\nu P$$

and

$$E_\mu^\nu = (1/4\pi)(-F^{\nu\alpha}F_{\mu\alpha} + \tfrac{1}{4}\delta_\mu^\nu F^{\alpha\beta}F_{\alpha\beta}) \tag{5}$$

where P is the internal pressure, ρ, σ are the densities of matter and charge respectively and V^{ν} is the velocity vector of matter.

The static condition is given by $V^i = 0$ and $V^0 = (g_{00})^{-1/2}$. We shall assume the field to be purely electrostatic, ie, $F_{ik} = 0$ and $F_{0k} = \phi_{,k} \equiv \phi_k$ where ϕ is the electrostatic potential.

The field equations take the form

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = 8\pi P - E \tag{6}$$

$$e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right) = 8\pi P + E \tag{7}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = 8\pi\rho + E \tag{8}$$

where

$$E = -F^{41}F_{41} \tag{9}$$

and

$$4\pi\sigma = \left(\frac{dF^{41}}{dr}+\frac{2}{r}F^{41}+\frac{\lambda'+\nu'}{2}F^{41}\right)e^{\nu/2}. \tag{10}$$

We can write the equations (6)–(8) in the form

$$8\pi P = \frac{e^{-\lambda}}{2}\left(\frac{3}{2}\frac{\nu'}{r}+\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}-\frac{\lambda'}{2r}+\frac{1}{r^2}\right)-\frac{1}{2r^2} \tag{11}$$

$$E = \frac{e^{-\lambda}}{2}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}+\frac{\nu'^2}{4}-\frac{\nu'}{2r}-\frac{\lambda'}{2r}-\frac{1}{r^2}\right)+\frac{1}{2r^2} \tag{12}$$

$$8\pi\rho = e^{-\lambda}\left(\frac{5}{4}\frac{\lambda'}{r}-\frac{\nu''}{4}+\frac{\lambda'\nu'}{8}-\frac{\nu'^2}{8}+\frac{\nu'}{4r}-\frac{1}{2r^2}\right)+\frac{1}{2r^2}. \tag{13}$$

We have four equations (6)–(8) and (10) and six variables (ρ, E, P, λ, ν, σ). Hence we have two variables free. We take λ and ν as the two free variables. Considering singularity-free conditions at $r \to 0$, the field equations (11)–(13) lead us to take the following simple forms for λ and ν:

$$\lambda = Ar^2 \tag{14}$$

$$\nu = Br^2 + C \tag{15}$$

where A, B, C are arbitrary constants. Using equations (14) and (15) in equations (6)–(8) and (10) we get

$$16\pi P = e^{-Ar^2}\left(4B - A + B(B-A)r^2 + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{16}$$

$$2E = e^{-Ar^2}\left(B(B-A)r^2 - A - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{17}$$

$$16\pi\rho = e^{-Ar^2}\left(5A - B(B-A)r^2 - \frac{1}{r^2}\right) + \frac{1}{r^2} \tag{18}$$

$$4\pi\sigma = \left(\frac{dF^{41}}{dr}+\frac{2}{r}F^{41}+(A+B)rF^{41}\right)e^{(Br^2+C)/2} \tag{19}$$

where

$$F^{41} = \left[e^{-2Ar^2-Br^2-C}\left(\tfrac{1}{2}B^2r^2-\frac{AB}{2}r^2-\frac{A}{2}-\frac{1}{2r^2}\right)+\frac{e^{-Ar^2-Br^2-C}}{2r^2}\right]^{1/2}. \tag{20}$$

3. Central and boundary conditions

At $r = 0$, we have from equations (16)–(20)

$$16\pi P_0 = 4B-2A \tag{21}$$

$$E_0 = 0 \tag{22}$$

$$16\pi\rho_0 = 6A \tag{23}$$

$$4\pi\sigma_0 = \tfrac{3}{2}[B^2+(A-B)^2]^{1/2}. \tag{24}$$

For P_0 and ρ_0 to be positive, we have respectively

$$2B \geqslant A \tag{25}$$

$$A \geqslant 0. \tag{26}$$

Further for $\rho_0 \geqslant 3P_0$

$$A \geqslant B. \tag{27}$$

From conditions (25) and (27)

$$2B \geqslant A \geqslant B. \tag{28}$$

Next we take up the conditions to be satisfied at $r = r_1$ (boundary).
(i) $P_1 = 0$. From equation (16) we get

$$e^{-Ar_1^2}\left(4B-ABr_1^2+Br_1^2-A+\frac{1}{r_1^2}\right)-\frac{1}{r_1^2} = 0. \tag{29}$$

This equation has a unique solution r_1 and since the pressure is positive at $r = 0$ the pressure must remain positive for all $r < r_1$.

(ii) $E_1 = Q^2/r_1^4$,

where Q is the total charge of the sphere. From equations (17) and (29)

$$e^{-Ar_1^2}\left(2Br_1+\frac{1}{r_1}\right) = \frac{1}{r_1}-\frac{Q^2}{r_1^3}. \tag{30}$$

Also equations (17) and (29) give the following condition for Q^2 being positive:

$$r_1^2 < \frac{2B-A}{B(A-B)}. \tag{31}$$

The condition (28) shows that the right-hand side of this inequality is positive.

We can also see that E is positive throughout the sphere. From equation (17) we have

$$2E = e^{-Ar^2}\left(\tfrac{1}{2}[B^2+(A-B)^2]r^2+\frac{A^3r^4}{3!}+\frac{A^4r^6}{4!}+\ldots\right). \tag{32}$$

Obviously the right-hand side is positive.
(iii) $\rho_1 \geqslant 0$. From equations (18) and (29)

$$A+B \geqslant 0. \tag{33}$$

It appears from the condition (33) that ρ_1 cannot be zero at $r = r_1$. For, in that case $A = B = 0$ since both A and B are positive by (25) and (26). This makes $\rho = P = E = \sigma = 0$ throughout the sphere which means non-existence of the sphere itself.

We can easily see that ρ is positive throughout the sphere. From equation (18) we have

$$16\pi\rho = e^{-Ar^2}\left(6A+B(A-B)r^2+\frac{A^2r^2}{2!}+\frac{A^3r^4}{3!}+\ldots\right). \tag{34}$$

Evidently the right-hand side is positive.
(iv) $\lambda_1+\nu_1 = 0$. Using equations (14) and (15) we have

$$Ar_1^2+Br_1^2+C = 0. \tag{35}$$

Equation (35) shows that C is negative since A, B and r_1^2 are all positive.
(v) $e^{-\lambda_1} = 1-(2M/r_1)+(4\pi Q^2/r_1^2)$, where M is the mass of the sphere. Using equation (14) we get

$$e^{-Ar_1^2} = 1-\frac{2M}{r_1}+\frac{4\pi Q^2}{r_1^2}. \tag{36}$$

The central and boundary conditions fix up the constants of the solution.

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